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Numerical technique to solve dynamical system involving fuzzy parameters

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Abstract

This paper investigates system of differential equations with fuzzy parameters and fuzzy initial condition i.e. given by $\tilde{X}(t) = \tilde{f}(t,\tilde{X})$, $\tilde{X}(t_0) = \tilde{X}_0$ for numerical solution. Here $\tilde{f}(t,\tilde{X})$ can be nonlinear or it can be linear of the form $\tilde{A}\otimes\tilde{X} + \tilde{B}$, where $\tilde{A}_{n\times n}$ and $\tilde{B}_{n\times 1}$ be some matrices with all entries as fuzzy number. In this paper, we propose the numerical technique which is based on approximation of Hukuhara difference, for both kind of dynamical systems (linear and nonlinear). In dynamical system, uncertainty of possibilistic type can be realized efficiently using fuzzy parameters and such systems are mathematical models for various application in varied domains. For such systems, we get the scheme for existence of solution and its convergence. Lastly, illustrative examples are solved by using proposed scheme and compared with crisp solution.

Keywords: Fuzzy parameters, Fuzzy number,Hukuhara differentiability, Linear and nonlinear fuzzy dynamical system.

1.INTRODUCTION

Since Zadeh's first paper on fuzzy set in 1965, there has been lots of development in various field of fuzzy set theory. Most of the real-world problem can be modelled as a dynamical system but while modelling such problems, uncertainty mayoccur in estimating parameters or/and initial condition, so to reduce such kind of possibilistic uncertainties in dynamical system, fuzzy theory is widely used.

The concept of fuzzy derivative was first introduced by Chang and Zadeh[1] in 1972. Dubois and Prade[2] defined derivative based on extension principle in 1982.Puri and Ralescue[3] introduced H-derivative of fuzzy-number-valued function.Seikkala[4] and Kaleva [5] first simultaneously solved the fuzzy initial value problem with fuzzy initial condition.Kandel and Byatt applied this theory to fuzzy dynamical system. The basic and most popular approach to solve Fuzzy Differential Equation (FDE) is Hukuhara Differentiability. But Hukuhara differentiability has disadvantage that,FDE requires increasing length of support. To deal with this problem H$\ddot{u}$llermeier[6]gave FDE as family of differential inclusions but the main problem with differential inclusion is not having derivative of fuzzy number valued functions. Bede and Gal [7]proposed the alternative of Hukuhara differentiability, i.e. strongly generalized differentiability (GH-differentiability).This concept is based on four forms of lateral derivatives. The disadvantage of this differentiability is that FDE can have several solutions, locally have two, so choose that one which reflects practically possible solution for modelled problem.

At times, it may be difficult or not possible to find exact solution of FDE, also the real-life applications may have only the observations for the dynamical processes involving imprecisionthen for solving such problems use of numerical methods becomes inevitable. Numerical method for solving FDE with fuzzy initial condition given by many authors[8],

[9],[10],[11],[12],[13],[14],[15],[16],[17],[18], [19], [20], [21] and analytical technique used in [33]. In [25], [26], [27], [28], [29] authors solved system of fuzzy differential equations with/or fuzzy parameters and fuzzy initial condition.

In this paper, we propose numerical scheme by approximating the Hukuhara difference and proposed scheme is verified by solving two examples, first example is based on toxic discharge in river by mills [31] and second is based on modelling of spread of infection disease [32].

The paper contains 5 sections,insection 2, basic definitions regarding fuzzy set theory are given, in section 3, numerical scheme is proposed and in last sections examples and conclusion are given.

2.PRELIMINARIES

Let $\mathcal{P}(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $\mathcal{P}(R^n)$ as usual. Let $A, B \in R^n$. The distance between A and B is defined by Housdorff metric.

$$d(A,B) = max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

$\|\cdot\|$ denotes usual norm in R^n. $(\mathcal{P}(R^n), d)$ is a complete metric space.

Let,
$E^n = \{\tilde{u}: R^n \to [0\ 1] /\ \tilde{u}$ satisfies following properties$\}$
- $\tilde{u}$ is normal i.e there exist $x_0 \in R^n \tilde{u}(x_0) = 1$.
- $\tilde{u}$is a fuzzy convex.
- $\tilde{u}$is upper semicontinuous.
- $[\tilde{u}]^0 = \overline{\text{supp}\ (\tilde{u}) = \{x \in R^n / \tilde{u}(x) \geq 0\}}$ is compact.

For $0 < \alpha \leq 1$, denote$\tilde{u}^\alpha = \{ x \in R^n / \tilde{u}(x) \geq \alpha\}$ then from the above properties follows, α level sets $\tilde{u}^\alpha \in \mathcal{P}(R^n) \forall \alpha \in [0\ 1]$

Fuzzy number in parametric form

A fuzzy number in parametric form is an order pair of the form $\tilde{u}^\alpha = (\underline{u}(\alpha), \overline{u}(\alpha)) = [\underline{u}\overline{u}]$
where$0 \leq \alpha \leq 1$ satisfying following condition:
- $\underline{u}$is bounded left continuous increasing function in $[0\ 1]$.
- $\overline{u}$ is bounded right continuous increasing function in$[0\ 1]$.
- $\underline{u} \leq \overline{u}$

Triangular Fuzzy Number

Triangular fuzzy number is defined as with three points (l, m, n),

$$\tilde{u}(x) = \begin{cases} \dfrac{x-l}{m-l} & l \leq x \leq m \\ \dfrac{n-x}{n-m} & m \leq x \leq n \\ 0 & x \geq n \end{cases}$$

$\tilde{u}(x)$ is membership function.

Hausdorff Distance

Let $d: E^n \times E^n \to R_+ \cup \{0\}$,

$$d(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0\ 1]} \max|\underline{u} - \underline{v}, \overline{u} - \overline{v}|$$

is Hausdorff distance between two fuzzy numbers *u* and *v*. (E^n, d) is a complete metric space[3].

Fuzzy Operation

For$\tilde{u}, \tilde{v} \in E^n$ and $\lambda \in R$ the sum $\tilde{u} + \tilde{v}$and the product $\lambda \otimes \tilde{u}$ is defined as

$$[\tilde{u} + \tilde{v}]^\alpha = [\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}],\ [\lambda \otimes u]^\alpha = \lambda[\underline{u}, \overline{u}] = [\lambda.\overline{u}, \lambda.\underline{u}] \forall \alpha \in [0\ 1].$$

$$[\tilde{u} \otimes \tilde{v}]^\alpha = [\min(\underline{u}\overline{v}, \underline{v}\overline{u}, \underline{u}\underline{v}, \overline{u}\overline{v}), \max(\underline{u}\overline{v}, \underline{v}\overline{u}, \underline{u}\underline{v}, \overline{u}\overline{v})] \forall \alpha \in [0\ 1]$$

Continuity of Fuzzy function

If $f: R \times E^n \to E^n$then $f$is continuous at point (t_0, x_0) provided that for any fixed number $\alpha \in [0\ 1]$ and any $\epsilon > 0, \exists\ \delta(\epsilon, \alpha) s.t\ d([\tilde{f}(t, \tilde{x})]^\alpha, [\tilde{f}(t_0, \tilde{x}_0)]^\alpha) < \epsilon$ whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([\tilde{x}]^\alpha, [\tilde{x}_0]^\alpha) < \delta(\epsilon, \alpha) \forall t \in R$ and$\tilde{x} \in E^n$.(Song and Wu as in [22])

Hukuhara Derivative

As in [3], for a fuzzy mapping$\tilde{f}: I \times E^n \to E^n$ and $t_0 \in I$, $\tilde{f}$ is said to be differentiable at $t_0 \in I$, if there exist an element $\tilde{f}'(t_0) \in E^n$such that for all $h > 0$ sufficiently small$\exists\ \tilde{f}(t_0 + h) \ominus \tilde{f}(t_0), \tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \tilde{f}'(t_0)$$

Let $x, y \in E^n$. if there exists $z \in E^n$such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$. $x \ominus y \neq x + (-1)y$. In this paper "$\ominus$" stands for always H-difference.

Seikkala Derivative

This derivative follows if $[\dot{\underline{x}}(t), \dot{\overline{x}}(t)]$ are the $\alpha\ cut$ of any fuzzy numbers then $SD\widetilde{X}^\alpha(t)$ exists and is defined as, [4]

$$SD\widetilde{X}^\alpha(t) = [\dot{\underline{x}}(t), \dot{\overline{x}}(t)]$$

3.Fully Fuzzy Dynamical System

Consider a system of ODE in E^n with fuzzy coefficient,

$$\dot{\tilde{X}}(t) = \tilde{f}(t, \tilde{X}), \tilde{X}(t_0) = \tilde{X}_0 \quad (1)$$

where, $\tilde{f}: [t_0\ T] \times E^n \to E^n$ and

$$\tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_n \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}, \tilde{X}_0 = \begin{bmatrix} \tilde{x}_{10} \\ \tilde{x}_{20} \\ \vdots \\ \tilde{x}_{n0} \end{bmatrix}$$

Here each $\tilde{f}_i$ is Hukuhara differentiable, $\forall\ i = 1,2,3 \ldots..$ $\tilde{f}$can be linear or nonlinear so we consider both cases.

Case 1:$\tilde{f}$ is linear function

System (1) is given as,

$$\dot{\tilde{X}}(t) = \tilde{A} \otimes \tilde{X} + \tilde{B} \tilde{X}(t_0) = \widetilde{X_0} \quad t \in [t_0\ T] \quad (2)$$

Where,$\tilde{A} = (\widetilde{a_{ij}}) \in E$and $\tilde{B} = (\widetilde{b_i}) \in E$, where $i = 1,2,3 \ldots \ldots \ldots, j = 1,2,3 \ldots \ldots ..$
$\alpha -$ cut of $\tilde{A}$,$\tilde{X}$, $\tilde{B}$ and $\tilde{X}_0$ are given as,

$$\tilde{A}^\alpha = [\underline{A}\overline{A}], \tilde{X}^\alpha = [\underline{X}\overline{X}], \tilde{B}^\alpha = [\underline{B}\overline{B}] \text{ and } \widetilde{X_0}^\alpha = [\underline{X_0}\overline{X_0}].$$

After taking $\alpha\ cut$ of system (2), it is converted into system of ordinary differential equations, Refer [15,23,24].

$$[\dot{\underline{X}}\dot{\overline{X}}] = [\underline{A}\overline{A}][\underline{X}\overline{X}] + [\underline{B}\overline{B}]; [\underline{X_0}\overline{X_0}]$$

$$\dot{\underline{X}}(t) = \min(\underline{A}\underline{X}, \underline{A}\overline{X}, \underline{X}\overline{A}, \overline{A}\overline{X}) + \underline{B}\underline{X}(0) = \underline{X_0}$$

$$\dot{\overline{X}}(t) = \max(\underline{A}\underline{X}, \underline{A}\overline{X}, \underline{X}\overline{A}, \overline{A}\overline{X}) + \overline{B}\,\overline{X}(0) = \overline{X_0}$$

Above system can be written in matrix form as given below,

$$\begin{bmatrix} \dot{\underline{X}}(t) \\ \dot{\overline{X}}(t) \end{bmatrix} = \begin{bmatrix} \underline{A} & 0 \\ 0 & \overline{A} \end{bmatrix} \begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \overline{B} \end{bmatrix}; \quad (3)$$

$\underline{X}(0) = \underline{X_0}$ and $\overline{X}(0) = \overline{X_0}$

Now by approximating Hukuhara difference of the left side of system (3) at $t = t_k$,

$$\begin{bmatrix}\underline{X}_{k+1}\\ \overline{X}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{B}\\ \overline{B}\end{bmatrix}; \quad (4)$$

$$\underline{X}(0) = \underline{X_0} \text{ and } \overline{X}(0) = \overline{X_0}$$

where $k = 0, 1, 2, \dots\dots\dots$

For solving purpose, system (4) can be given as component wise,

$$\begin{bmatrix}\underline{x}^1{}_{k+1}\\ \underline{x}^2{}_{k+1}\\ \vdots\\ \underline{x}^n{}_{k+1}\end{bmatrix} = \begin{bmatrix}(I + h\underline{a}_{11}) & \cdots & \underline{a}_{1n}\\ \vdots & \ddots & \vdots\\ \underline{a}_{n1} & \cdots & (I + h\underline{a}_{nn})\end{bmatrix}\begin{bmatrix}\underline{x}^1{}_k\\ \underline{x}^2{}_k\\ \vdots\\ \underline{x}^n{}_k\end{bmatrix} + h\begin{bmatrix}\underline{b}_1\\ \underline{b}_2\\ \vdots\\ \underline{b}_n\end{bmatrix}$$

$$\begin{bmatrix}\overline{x}^1{}_{k+1}\\ \overline{x}^2{}_{k+1}\\ \vdots\\ \overline{x}^n{}_{k+1}\end{bmatrix} = \begin{bmatrix}(I + h\overline{a}_{11}) & \cdots & \overline{a}_{1n}\\ \vdots & \ddots & \vdots\\ \overline{a}_{n1} & \cdots & (I + h\overline{a}_{nn})\end{bmatrix}\begin{bmatrix}\overline{x}^1{}_k\\ \overline{x}^2{}_k\\ \vdots\\ \overline{x}^n{}_k\end{bmatrix} + h\begin{bmatrix}\overline{b}_1\\ \overline{b}_2\\ \vdots\\ \overline{b}_n\end{bmatrix}$$

Existence and uniqueness of solution for system such as (4) is given in [30].

Theorem (1):
Let $\underline{X}(t), \overline{X}(t)$ be the exact solution of system (3) and $\underline{X}_{k+1}(t), \overline{X}_{k+1}(t)$ be the sequences of numerical solution defined by the system (4)converges to the exact solution of system (3).

Proof: It is sufficient to show,

$$\lim_{k\to\infty} \underline{X}_{k+1}(t) = \underline{X}(t)$$
$$\lim_{k\to\infty} \overline{X}_{k+1}(t) = \overline{X}(t)$$

By proposed scheme in system (4) is given as,

$$\begin{bmatrix}\underline{X}_{k+1}\\ \overline{X}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{B}\\ \overline{B}\end{bmatrix}$$

with initial conditions, $\underline{X}(0) = \underline{X_0}$ and $\overline{X}(0) = \overline{X_0}$,
From system (3), writing the equivalent discrete form,

$$\begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} = \begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} + h\begin{bmatrix}\underline{B}\\ \overline{B}\end{bmatrix} \quad (5)$$

By subtractingSystem (5) from system (4),

$$\begin{bmatrix}\underline{X}_{k+1} - \underline{X}\\ \overline{X}_{k+1} - \overline{X}\end{bmatrix} = \begin{bmatrix}\underline{X}_k - \underline{X}\\ \overline{X}_k - \overline{X}\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}_k - \underline{X}\\ \overline{X}_k - \overline{X}\end{bmatrix}$$

By using Error term,

$$\begin{bmatrix}\underline{E}_{k+1}\\ \overline{E}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{E}_k\\ \overline{E}_k\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{E}_k\\ \overline{E}_k\end{bmatrix}$$

For solving purpose, we can write above system as following,

$$\underline{E}_{k+1} = \underline{E}_k + h\underline{A}\underline{E}_k$$
$$\overline{E}_{k+1} = \overline{E}_k + h\overline{A}\overline{E}_k$$

Now by backward substitution,

$$\underline{E}_{k+1} = (I + h\underline{A})^{(k+1)} \underline{E}_0$$

$\underline{A}$ is nonsingular matrix so $(I + h\underline{A})$ is also nonsingular matrix so the solution of system exists.
For the convergence of system (4),

$$|(I + h\underline{A})| < 1$$

Take $\underline{E}_0 = 0, \underline{E}_{k+1} \to 0$
That is,

$$\lim_{k\to\infty} \underline{X}_{k+1}(t) = \underline{X}(t)$$

Similarly,

$$\lim_{k\to\infty} \overline{X}_{k+1}(t) = \overline{X}(t)$$

Case 2: $\tilde{f}$ *is nonlinear function*

Consider a system of ODE in E^n with fuzzy coefficient,

$$\dot{\tilde{X}}(t) = \tilde{f}(t, \tilde{X}), \tilde{X}(t_0) = \tilde{X}_0 \quad (6)$$

where, $\tilde{f}: [t_0 \ T] \times E^n \to E^n$ and

$$\tilde{f} = \begin{bmatrix}\tilde{f}_1\\ \tilde{f}_2\\ \vdots\\ \tilde{f}_n\end{bmatrix}, \tilde{X} = \begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \vdots\\ \tilde{x}_n\end{bmatrix}, \tilde{X}_0 = \begin{bmatrix}\tilde{x}_{10}\\ \tilde{x}_{20}\\ \vdots\\ \tilde{x}_{n0}\end{bmatrix}$$

$\tilde{X}(t)$ is Hukuhara differentiable or Seikkala differentiable then,

$$\dot{\tilde{X}}^\alpha(t) = \left[\underline{\dot{X}}\dot{\overline{X}}\right], \tilde{f}^\alpha = \left[\underline{f}\overline{f}\right] \text{and } \widetilde{X_0}^\alpha = \left[\underline{X_0}\overline{X_0}\right]$$

Taking α *cut* of system (6) and we get,

$$\left[\underline{\dot{X}}\dot{\overline{X}}\right] = \left[\underline{f}(t, \underline{X}, \overline{X}), \overline{f}(t, \underline{X}, \overline{X})\right]$$

with initial condition $\left[\underline{X_0}\overline{X_0}\right]$

Above system can be presented in form of system of ordinary differential equations [15,23,24]

$$\left.\begin{array}{l}\underline{\dot{X}}(t) = \underline{f}(t, \underline{X}, \overline{X}); \ \underline{X}(0) = \underline{X_0}\\ \dot{\overline{X}}(t) = \overline{f}(t, \underline{X}, \overline{X}); \overline{X}(0) = \overline{X_0}\end{array}\right\} \quad (7)$$

Now by the proposed scheme,

$$\left.\begin{array}{l}\underline{X}_{k+1} = \underline{X}_k + h\underline{f}(t, \underline{X}_k, \overline{X}_k); \underline{X}(0) = \underline{X_0}\\ \overline{X}_{k+1} = \overline{X}_k + h\overline{f}(t, \underline{X}_k, \overline{X}_k); \overline{X}(0) = \overline{X_0}\end{array}\right\} \quad (8)$$

Lemma 1: (Refer [8]), Letthe sequence of numbers $\{e_n\}_{n=0}^N$ satisfy

$$|e_{n+1}| \le A|e_n| + B, \qquad 0 \le n \le N - 1$$

For the given positive constants A and B. Then

$$|e_n| \le A^n|e_0| + B\frac{A^n - 1}{A - 1},$$

Theorem (2): $\underline{f}(t, \underline{X}, \overline{X})$, $\overline{f}(t, \underline{X}, \overline{X}) \in E^n$, their partial derivatives $\frac{\partial \underline{f}(t, \underline{X}, \overline{X})}{\partial \underline{X}}$ and $\frac{\partial \overline{f}(t, \underline{X}, \overline{X})}{\partial \overline{X}}$ bounded and Lipschitz in E^n. Let $\underline{X}$, $\overline{X}$ both are exact solution of system (7) and $\underline{Y}$, $\overline{Y}$ both are numerical solution of system (8) then numerical solution converges to the exact solution uniformly.

Proof: Define error in nth term,

$$e_n = \underline{X}(t_n) - \underline{Y}(t_n)$$
$$e_{n+1} = \underline{X}(t_{n+1}) - \underline{Y}(t_{n+1})$$

By using Taylor's expansion and neglecting higher terms,

$$e_{n+1} = [\underline{X}(t_n) + h\underline{X}'(t_n) + \frac{h^2}{2}\underline{X}''(t_n + \theta h) - \left(\underline{Y}(t_n) + h\underline{f}\left(t, \underline{Y}(t_n), \overline{Y}(t_n)\right)\right)]$$

$$e_{n+1} = \left(\underline{X}(t_n) - \underline{Y}(t_n)\right) + h(\underline{X}'(t_n) - \underline{f}\left(t, \underline{Y}(t_n), \overline{Y}(t_n)\right) + \frac{h^2}{2}\underline{X}''(t_n + \theta h)$$

By taking modulus and using Lipschitz condition,

$$|e_{n+1}| \le |e_n|(1 + hL) + N\frac{h^2}{2}$$

Computing in backward manner we get at k^{th} step, $k < n$

$$|e_n| \le |e_{n-1}| + hL|e_{n-1}| + N\frac{h^2}{2}$$

$$|e_{k-1}| \le \left[|e_{k-2}|(1 + hL) + N\frac{h^2}{2}\right](1 + hL) + N\frac{h^2}{2}$$

$$|e_1| \le |e_1| + hL|e_1| + N\frac{h^2}{2}$$

Applying lemma (1) we get,

$$|e_n| \le |e_0|(1 + hL)^n + N\frac{h^2}{2}\frac{((1 + hL)^n - 1)}{hL}$$

Let $A = 1 + hL$ and $B = N\frac{h^2}{2}$

$$|e_n| \le |e_0|(A)^n + B\frac{((A)^n - 1)}{A - 1}$$

$$|e_n| \le |e_0|(1 + hL)^n + N\frac{h^2}{2}\frac{((1 + hL)^n - 1)}{hL}$$

Suppose $e_0 = 0$ and $(1 + hL) \le (\exp)^{hL}$

$$|e_n| \le N\frac{h^2}{2}\frac{((\exp)^{nhL} - 1)}{hL}$$

$$|e_n| \le N\frac{h^2}{2}\frac{((\exp)^{nhL} - 1)}{hL}$$

which is valid for $0 \le t_n = nh \le T \rightarrow$ stability, finally,

$$\le N\frac{h^2}{2}\frac{((\exp)^{TL} - 1)}{hL}$$

as $h \rightarrow 0$,

$$|e_n| \rightarrow 0$$

$$|e_n| \le N\frac{h^2}{2}\frac{((\exp)^{TL} - 1)}{hL} = N\frac{h}{2}\frac{((\exp)^{TL} - 1)}{L} = O(h)$$

This establishes the linear convergence of proposed the iterative method.

Similarly, we can show linear convergence for $\overline{f}(t, \underline{X}, \overline{X})$.

4.NUMERICAL EXAMPLE

In this section, two examples to illustrate the use of proposed scheme from environmental science and biological science.

Water Pollution by Industrial Wastage

Contamination of lakes and rivers is common problem. The problem of predicting the impact of a toxic waste discharge into lakes and rivers is important for environmental sciences. The mathematical model for such a dynamical system is given as the system of ordinary linear differential equations, while modelling, exact estimation of quantity of toxic in discharge and fluid flow may not be possible so the considering the involved parameters represented by fuzzy numbers gives better realization.

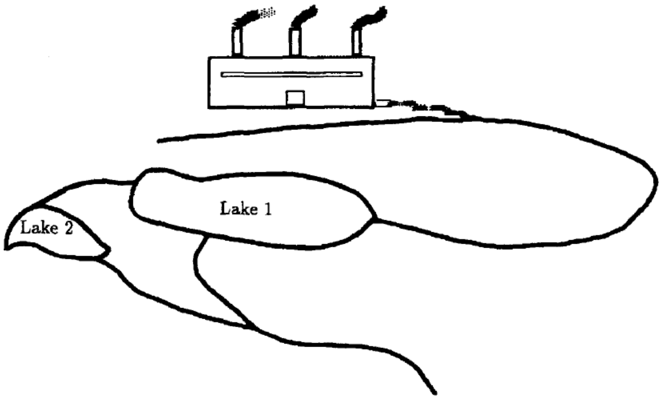

Figure 1 Lake and factory model

This dynamical system is taken from [31], we consider two lakes as shown in Figure 1 above, with following assumptions:

The toxic material leaks into the river at constant rate for one day. We assume that the toxic material mixes quickly and thoroughly with lake waters as it enters each lake. The toxic discharge will also have an impact on the second lake. Let water of lake 1 barely trickles into lake 2, suggests contamination of lake 2 should be minimal. Final differential equations are formed as below,

$$\frac{d\tau_1}{dt} = r(t) - \frac{i_1}{v_1}\tau_1$$
$$\frac{d\tau_2}{dt} = \frac{i_2}{v_1}\tau_1 - \frac{o_2}{v_2}\tau_2$$

Putting numerical value,

$$\frac{d\tau_1}{dt} = \widetilde{100} - \widetilde{0.5}\tau_1$$
$$\frac{d\tau_2}{dt} = \widetilde{0.25}\,\tau_1 - \tilde{1}\tau_2$$

with initial conditions:

$$\tau_1^0(0) = \widetilde{50} = (40,50,60) = (40 + 10\alpha, 60 - 10\alpha),$$
$$\tau_2^0(0) = \widetilde{10} = (5,10,15) = (5 + 5\alpha, 15 - 5\alpha),$$

and $h = 0.1$,

$$\widetilde{100} = (90,100,110) = (90 + 10\alpha, 110 - 10\alpha),$$
$$\widetilde{0.5} = (0.4,0.5,0.6) = (0.4 + 0.1\alpha, 0.6 - 0.1\alpha)$$
$$\widetilde{0.25} = (0.20, .25,30) = (0.20 + .05\alpha, .30 - .05\alpha),$$
$$\tilde{1} = (0.5,1,1.5) = (0.5 + 0.5\alpha, 1.5 - .5\alpha)$$

Applying the proposed scheme and we get,

$$\tau_1^1(t) = [-0.1\,\alpha^2 + 12.2\,\alpha + 45.4, -0.1\,\alpha^2 - 11.8\,\alpha + 69.4]$$

$$\tau_2^1(t) = [-0.2\alpha^2 + 6.9\alpha + 3.55, -0.2\,\alpha^2 - 6.1\,\alpha + 16.55]$$

The other iteration values at different times for τ_1 and τ_2 are as shown in Figure 2 and Figure 3 respectively.

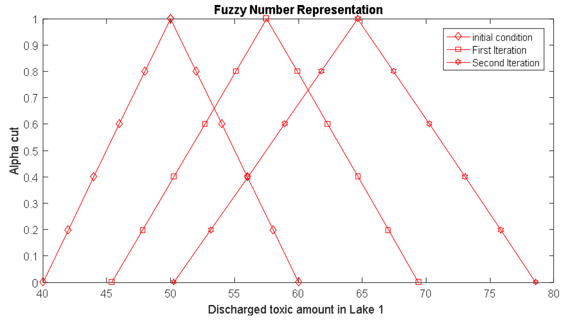

Figure 2 Fuzzy number representation for discharged toxic in Lake 1

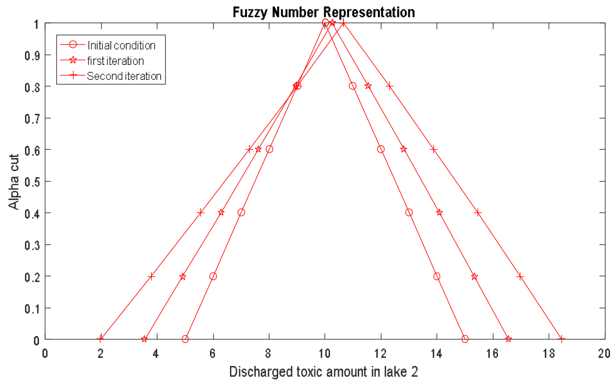

Figure 3 Fuzzy number representation for discharged toxic in Lake 2

Table 1 Number of iterations for amount of toxic discharge in lakes

Time	τ_1 (toxic discharge amount in lake1 as a fuzzy triplet)	Crisp solution for τ_1	τ_2(toxic discharge amount in lake 2 as a fuzzy triplet)	Crisp solution for τ_2
t = 0	(40, 50, 60)	50	(5, 10, 15)	10
t = 0.1	(37.5, 57.5, 69.4)	57.5	(3.55, 10.25, 16.55)	10.25
t = 0.2	(34.23, 64.625, 78.90)	64.625	(1.975, 10.6625, 18.4545)	10.6625

We can see from the Table 1, core value of fuzzy solution matches with crisp solution by same numerical technique and support increases with time.

Spread of Infectious Disease Model

This application is taken from [32]. There is a total population of people is N and certain contagious disease infecting the people. This population is divided into three parts, $x(t)$=those uninfected but may be infected, $y(t)$=those presently infected and may spread the disease, $z(t)$=already had the disease are dead, recovered and immune or cannot spread the disease.

So, $N = x(t) + y(t) + z(t)$

The rate of transfer from x into y is directly proportional to xy. So, $\dot{x}(t) = -kx(t)y(t)$

The rate of transfer into y comes from x and the rate of transfer out of y goes to z which is proportional to y, so, $\dot{y}(t) = kx(t)y(t) - cy(t)$ where k and c are positive constants and they are need to be estimated. There is no need of third differential equation because we may get z by this relation,

$$N = x(t) + y(t) + z(t).$$
$$\dot{x} = -kxy$$
$$\dot{y} = kxy - cy$$
$$z(t) = N - (x + y)$$

with initial condition x_0, y_0 and $z_0 = 0$, $N = x_0 + y_0$

k & c depends on type of disease and season too so these parameters can be fuzzified.

Hence,

$$\tilde{k}^{\alpha} = (0.003, 0.005, 0.007) = (0.003 + 0.002\alpha, 0.007 - 0.002\alpha)$$
$$\tilde{c}^{\alpha} = (0.6, 0.9, 1.2) = (0.6 + 0.3\alpha, 1.2 - 0.3\alpha)$$
$$\tilde{x}^{\alpha}{}_0 = (920, 950, 980) = (920 + 30\alpha, 980 - 30\alpha)$$
$$\tilde{y}^{\alpha}{}_0 = (20, 50, 80) = (20 + 30\alpha, 80 - 30\alpha)$$ and $N = 1000$

Applying proposed numerical technique, we get first iteration which is given below,

$$[\underline{x_1}\overline{x_1}] = [0.18\alpha^3 - 6.99\alpha^2 + 67.94\alpha + 865.12, -0.18\alpha^3 - 5.91\alpha^2 - 42.14\alpha + 974.48]$$

$$[\underline{y_1}\overline{y_1}] = [0.18\alpha^3 + 5.01\alpha^2 + 48.14\alpha + 15.92, -0.18\alpha^3 + 6.90\alpha^2 - 70.34\alpha + 133.68]$$

For next iterations, Figures 4 & 5 are given below,

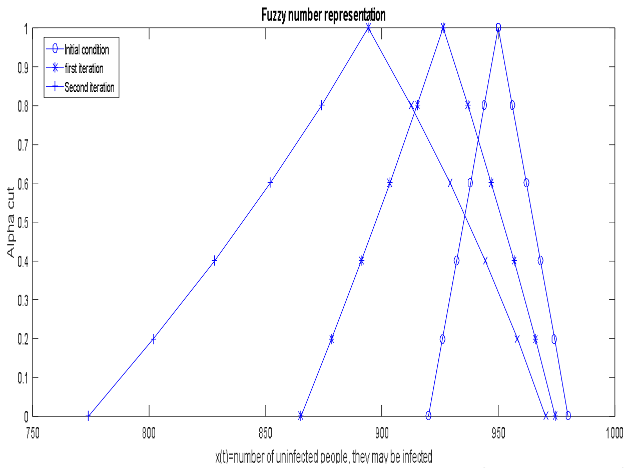

Figure 4 Fuzzy number representation for number of uninfected people

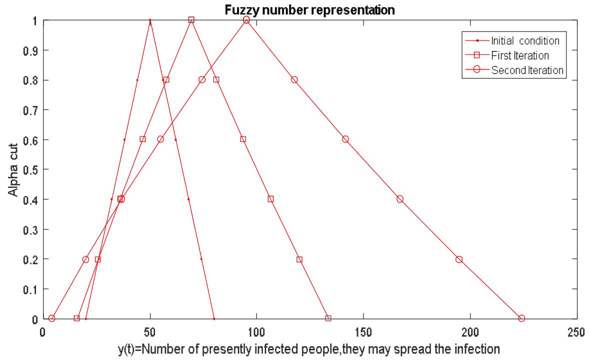

Figure 5 Fuzzy number representation for number of infected people

Table 2 Number of iterations for uninfected and infected people

Time	**x (Uninfected people but may be infected as a fuzzy triplet)**	**Crisp solution for x**	**y (Infected people and may spread the disease as a fuzzy triplet)**	**Crisp solution for y**
t = 0	(920, 950, 980)	950	(20, 50, 80)	50
t = 0.1	(865.12, 926.25, 974.48)	926.25	(15.92, 69.25, 133.68)	69.25
t = 0.2	(773.93, 894.178, 970.348)	894.178	(4.01, 95.08, 223.91)	95.08

From table 2, core value of fuzzy solution matches with crisp solution and support increases as time increases.The value of $z(t)$ can be calculated in each iteration by this relation,

$$z(t) = 1000 - x(t) - y(t).$$

Table 3 Number of iterations for already dead, recovered and immune people

Time	**z (already had the disease are dead, recovered and immune or cannot spread the disease)**
t=0.1	4.5
t=0.2	10.742

5.CONCLUSION

We have proposed numerical technique to solve fully fuzzy dynamical systems. Established convergence for the proposed scheme and applied it to two real world problems and compared the results with crisp solution. This technique can also be applied on fuzzy dynamical system under generalized differentiability, however, we may not get unique solution in some cases.

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