Numerical Integration of Probability Density Functions by Quadrature Method

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Abstract

Numerical solution of normal, gamma, exponential Weibull probability density function are proposed in this paper, based on quadrature method like Haar wavelet method are applied numerically to solve these integrals. the high accuracy the wide applicability of the Haar wavelet approach will be illustrated with numerical examples.

Keywords: Numerical Integration, Probability Density function, Haar wavelet.

1. INTRODUCTION

The problem is considered in this paper is the numerical integration of the probability density functions:

$$e^{\frac{-x^2}{2}}, \qquad x^m e^{\frac{-x^2}{2}}, \qquad \left(\frac{x}{\lambda}\right)^{m-1} e^{\left(\frac{x}{\lambda}\right)^m}, \qquad \lambda\, e^{-\lambda x}$$

with different parameters in (a, b), these integrals are arrived in various fields of science and engineering as biomechanics, economics, statistics, financial mathematics, signal processing, plasticity, queuing theory, heat and mass transfer, electrostatics, mechanical engineering, game theory, control theory, system simulation and modeling etc. various numerical methods have been widely used, such as midpoint rule, Newton cotes, Rom berg integration method, Gaussian quadrature etc. Numerical integration is expensive in computation time and require higher order of integration to obtain an accurate approximation, Numerical integration methods are used in evaluation of Feri - Dirac and Bose Einstein integrals that are frequently arise in quantum statistics [1-3], many numerical integration rules [4-8] are used function as polynomial, smooth, oscillating, rational and irrational expressions, numerical integration of multivariate normal and t probabilities are evaluated by integration technique are discussed in [9], Haar wavelets and hybrid function are used to evaluate highly oscillating and non oscillating integrals in [12-14] recently, the author in [11] has proposed discrete momentum method for obtaining an accurate numerical integration of normal, exponential, gamma and Weibull probability density functions with different parameters in (a, b). In this paper, Haar wavelets method are used to evaluate these integrals in a simple way with high accuracy and low time cost, the necessary program has been developed in computational software Maple 13. This paper is organized as follows. In Section 2, we describe the numerical integration formula of Haar wavelets method. In Section 3, we approximated the solution of probability density function with different parameters and resolution parameters, the result is compared to other approximations reported in the literature.

2. NUMERICAL INTEGRATION USING HAAR WAVELET METHOD

the discrete Haar function are generated by using scaling function $H_1(x)$:

$$H_1(x) = \begin{cases} 1, & if\ x \in [0,1) \\ 0, & elsewhere \end{cases} \tag{1}$$

and corresponding $H_2(x)$ can be expressed as

$$H_2(x) = \begin{cases} 1, & if\ x \in [0,\frac{1}{2}) \\ -1, & if\ x \in [\frac{1}{2},0) \\ 0, & elsewhere \end{cases} \tag{2}$$

the explicit form of $H_i(x)$ as

$$H_i(x) = \begin{cases} 2^{\frac{j}{2}}, & if\ x \in [\alpha,\beta\,) \\ -2^{\frac{j}{2}}, & if\ x \in [\,\beta,\gamma) \\ 0, & elsewhere \end{cases} \tag{3}$$

Where

$$\alpha = a + \frac{b(b-a)k}{m}, \qquad \beta = \ a + \frac{b(b-a)(k+0.5)}{m}$$

$$\gamma = a + \frac{b(b-a)(k+1)}{m}, m = 2^j\ for\ j = 0,1,2,---,J$$

and $k = 0,1,2,---,m-1$

indicates the level of resolution and k is the translation parameters the indexing in equation (3) is calculated as i = m+k+1, if the minimal value m = 1, k = 0, i = 2 the minimal value of i is 2m = 2j+1 with the collocation points $x_i = (j - 0.5)/2m$, j = 1,2,3,- - - , 2m

the discretized form of f(x) using Haar wavelets as

$$f(x) = \sum_{i=1}^{2m} a_i H_i(x) \tag{4}$$

$$\int_a^b f(x)dx = \sum_{i=1}^{2m} a_i \int_a^b H_i(x)\,dx = a_1(b-a) \tag{5}$$

to get the Haar coefficients a_1 are taken in (Islam and Aziz. 2010):

$$a_1 = \frac{1}{2M}\sum_{i=1}^{2m} f\left(a + \frac{(b-a)(k-0.5)}{2M}\right) \quad (6)$$

rearranging equation (5) as

$$\int_a^b f(x)dx = \frac{(b-a)}{2M}\sum_{i=1}^{2m} f\left(a + \frac{(b-a)(k-0.5)}{2M}\right) \quad (7)$$

3. NUMERICAL RESULTS

we have computed the numerical integration of probability density function with different parameters (a, b) by Haar wavelet method and compare our results with that of results given in reference [11], results are tabulated in Table.1-6. and also plotted relative error verses resolution parameter n in Figure.1, we observe that resolution parameter n increases and relative error decreases. the Haar wavelet method uses significantly less cpu time, this becomes more prominent to reach a desired accuracy.

Table 1: k1 = f(x) = $e^{\frac{-x^2}{2}}$; a = 0; b = 2 the exact value of the integral with 8 digits accuracy is 1.19628801

n	computed results	results given in [11] with M = 5, k = 5, N = 40	Abs. error
2	1.199085682	1.19628969	0.002797672
5	1.196738600		0.00045059
10	1.196400760		0.00011275
20	1.196316206		2.8196E-05
50	1.196292524		4.514E-06
100	1.196289141		1.131E-06
200	1.196288294		2.84E-07
400	1.196288084		7.4E-08

Table 2: k2 = f(x) = $e^{\frac{-x^2}{2}}$; a = 0; b = 3 the exact value of the integral with 8 digits accuracy is 1.24993045

n	computed results	results given in [11] with M = 5, k = 10, N = 40	Abs. error
2	1.250636175	1.24993063	0.000705545
5	1.250053457		0.000122827
10	1.249961566		3.0936E-05
20	1.249938246		7.616E-06
50	1.249931694		1.064E-06
100	1.249930757		1.27E-07
200	1.249930523		1.07E-07
400	1.249930464		1.4E-08

Table 3: k3 = f(x) = $x^m\, e^{\frac{-x^2}{2}}$; a = 0, b = 2, m = 3: the exact value of the integral with 5 digits accuracy is 1.18799

n	computed results	results given in [11] with M = 5, k = 10, N = 20	Abs. error
2	1.206014190	1.18799	0.01802419
5	1.188885647		0.000895647
10	1.188213557		0.000223557
20	1.188044671		5.4671E-05
50	1.187997323		7.323E-06
100	1.187990554		5.54E-07
200	1.187998864		8.864E-06
400	1.187998444		8.444E-06

Table 4: k4 = f(x) = $x^m\, e^{\frac{-x^2}{2}}$, a = 0, b = 4, m = 3 the exact value of the integral with 5 digits accuracy is 1.99396

n	computed results	results given in [11] with M = 5, k = 10; N = 20	Abs. error
2	1.98631746	1.99396	0.00764254
5	1.994216868		0.000256868
10	1.99406523		0.00010523
20	1.993989956		2.9956E-05
50	1.993966303		6.303E-06
100	1.993962835		2.835E-06
200	1.99396196		1.96E-06
400	1.993961747		1.747E-06

Table 5: k5 = f(x) = $\left(\frac{x}{\lambda}\right)^{m-1} e^{\left(\frac{x}{\lambda}\right)^m}$, a = 0, b = 4, m = 3 and λ = 4: the exact value of the integral with 5 digits accuracy is 0.84283

n	computed results	results given in [11] with M = 5, k = 10, N = 20	Abs. error
2	0.846810192	0.84283	0.003980192
5	0.843444314		0.000614314
10	0.842980930		0.00015093
20	0.842865747		3.57472E-05
50	0.842833544		3.5436E-06
100	0.842828945		1.0551E-06
200	0.842827795		2.2047E-06
400	0.842827507		2.4926E-06

Table 6: k6 = f(x) = $\lambda\, e^{-\lambda x}$, a = 0, b = 2 and λ = 3: the exact value of the integral with 5 digits accuracy is 0.99752

n	computed results	Exact value	Abs. error
2	0.9097965623	0.99752	0.087723438
5	0.9827140613		0.014805939
10	0.9937903394		0.003729661
20	0.9965866850		0.000933315
50	0.9973716358		0.000148364
100	0.9974838415		3.61585E-05
200	0.9975118964		8.1036E-06
400	0.9975189103		1.0897E-06

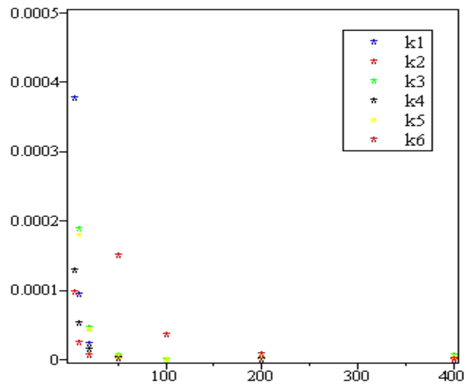

Figure 1 Relative error versus resolution parameter n

4. CONCLUSIONS

In this paper, Haar wavelets method is powerful tool for evaluating numerical integration of probability density function with different parameters (a, b) to get more accurate results with earlier computations. the simplicity and time cost are important factor of the method in many scientific and engineering fields.

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