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Generalized Hankel-Clifford transformation for a class of tempered ultradistributions

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Abstract: ***Present paper aims at investigating a theory supporting generalized Hankel-Clifford transformation on certain spaces of generalized functions. In an attempt to extend the transform to a space of tempered ultradistributions; definitions of classes of rapid descent ultradifferentiable functions is provided. Mappings involving various differential operators are shown to be continuous.***

Keywords: Generalized Hankel-Clifford transformation, tempered ultradifferentiable functions, differential operator

1 INTRODUCTION

In [1] the author studied Bessel Functions and their Applications to Physics. In [2] Ultradistributions: Structure Theorems and a Characterization have been studied. Integral transforms of generalized functions and their applications has been studied in [3]. In [6, 7] the author observed the Hankel-Clifford transformation on certain spaces of ultradistributions. In 2009, [5] author has analyzed generalized Hankel-type transformation for a class of tempered ultradistributions of Roumieu-type.

Malgonde [4] investigated the variant of the generalized Hankel-Clifford transform defined by

$$(h_{\alpha,\beta} f)(y) = F(y)$$

$$= \int_0^\infty (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) f(x)dx\ ,\ (\alpha-\beta) \geq -1/2$$

$$= y^{-\alpha-\beta}\int_0^\infty \mathrm{J}_{\alpha,\beta}(xy) f(x)dx\ ,\ (\alpha-\beta) \geq -1/2 \qquad (1.1)$$

where $\mathrm{J}_{\alpha,\beta}(x) = (x)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{x}\right)$, $J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions.

Certain results for certain differential operators and further, make use of a new defined differential operator and accordingly prove new relevant theorem are established. Hankel-type transform to spaces of tempered ultradistributions of Roumieu type have recently been discussed in [5].

2 ULTRADIFFERENTIABLE FUNCTIONS

The notation and terminology used in [4] will be continued. Sequences a_m and $b_q; m,q = 1,2,3,...$ are sequences of positive real numbers imposed by some of the following constraints as in [5].

Definition 2.1. Let $a > 0$ an arbitrary constant, $\alpha - \beta \in R$ and $p \in N$. Define the function of space ${}^pS_{\beta,a_m,A}$ as the collection of all complex valued smooth functions ψ defined on $I(0,\infty)$ such that set of all infinitely smooth functions satisfying

$$\left|x^m D^q\left(x^\beta \psi(x)\right)\right| \leq C_{q,\delta}(A+\delta)^m a_m \qquad (2.1)$$

for every $q \in N$ and $\delta > 0$.

Let $B > 0, b_q \geq 0$ an arbitrary constant $\alpha - \beta \in R$.

Define the function of space ${}^pS_\beta^{b_q,B}$ as the collection of all complex valued smooth

functions $\psi(x)$ defined on $I(0,\infty)$ such that set of all infinitely smooth functions satisfying

$$\sup_{x\in I}\left|x^m D^q\left(x^\beta \psi(x)\right)\right| \leq C_{m,\rho}(B+\rho)^q b_q \quad (2.2)$$

for every $m,q \in N$ and $\rho > 0$. $C_{m,\rho}$ are positive constant depending on ψ.

Let $A, B > 0$ and $a, b \geq 0$ an arbitrary constant $\alpha - \beta \in R$. Define the function of space ${}^pS_{\beta,a,A}^{b,B}$ as the collection of all complex-valued smooth functions $\psi(x)$ defined on $I(0,\infty)$ such that set of all infinitely smooth functions satisfying

$$\sup_{x\in I}\left|x^m D^q\left(x^\beta \psi(x)\right)\right| \leq C_{\delta,\rho}(A+\delta)^m (B+\rho)^q a_m b_q \qquad (2.3)$$

for every $m,q \in N$ and $\delta,\rho > 0$. $C_{\delta,\rho}$ are positive constant depending on ψ.

In view of Definition 2□1 ${}^pS_{\beta,a,A}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|_{q,\delta}=\sup_{\substack{x\in I\\ m\in N}}\frac{\left|x^m D^q\left(x^\beta\psi(x)\right)\right|}{(A+\delta)^m a_m} \qquad (2.4)$$

for every $m\in N$ and $\delta>0$, each $\|\ \|_{q,\delta}$ is a seminorm on ${}^pS_{\beta,a,A}$ and the collection $\Gamma=\left\{\|\ \|_{q,\delta}\right\}_{q\in N,\delta>0}$ is a multinorm because each $\|\ \|_{0,\delta}$ is a norm. Since the systems of seminorms Γ and $\Gamma_1=\left\{\|\ \|_{q,1/n}\right\}_{q,n\in N}$ are equivalent, the space ${}^pS_{\alpha,\beta,a,A}$ equipped with the topology generated by Γ_1, is a countable multinormed space.

${}^pS_\beta^{b,B}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|^{m,\rho}=\sup_{\substack{x\in I\\ q\in N}}\frac{\left|x^m D^q\left(x^\beta\psi(x)\right)\right|}{(B+\rho)^q b_q} \qquad (2.5)$$

for every $q\in N$ and $\rho>0$, each $\|\ \|^{m,\rho}$ is a seminorm on ${}^pS_\beta^{b,B}$ and the collection $\Gamma=\left\{\|\ \|^{m,\rho}\right\}_{m\in N,\,\rho>0}$ is a multinorm because each $\|\ \|^{0,\rho}$ is a norm in [8]. Since the systems of seminorms Γ and $\Gamma_1=\left\{\|\ \|^{m,1/n}\right\}_{m,n\in N}$ are equivalent, the space ${}^pS_\beta^{b,B}$ equipped with the topology generated by Γ_1, is a countable multinormed space.

${}^pS_{\beta,a,A}^{b,B}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|_\delta^\rho=\sup_{\substack{x\in I\\ m\in N\\ q\in N}}\frac{\left|x^m D^q\left(x^\beta\psi(x)\right)\right|}{(A+\delta)^m(B+\rho)^q a_m b_q} \qquad (2.6)$$

for every $m,q\in N$ and $\delta,\rho>0$, each $\|\ \|_\delta^\rho$ is a seminorm on ${}^pS_{\beta,a,A}^{b,B}$ and the collection $\Gamma_2=\left\{\|\ \|_\delta^\rho\right\}_{\delta,\,\rho>0}$ is a multinorm because each $\|\ \|^{0,\rho}$ is a norm. Since the systems of seminorms Γ_2 and $\Gamma_3=\left\{\|\ \|_{1/n}^{1/n}\right\}_{n\in N}$ are equivalent, the space ${}^pS_{\beta,a,A}^{b,B}$ equipped with the topology generated by Γ_3, is a countable multinormed space.

Inspired in the work of Roumieu [5], the dual spaces of ${}^pS'_{\beta,a,A}$, ${}^pS'^{b,B}_\beta$, ${}^pS'^{b,B}_{\beta,a,A}$ are called tempered ultradistributions of Roumieu-type. The generalized Hankel-Clifford transformation transform of the classes on the duals is called tempered ultradistributional generalized Hankel-Clifford transformation of Roumieu-type.

3 Operations of ${}^pS_{\beta,a_m,A}$, ${}^pS_\beta^{b_q,B}$ and ${}^pS_{\beta,a_m,A}^{b_q,B}$

The differential operators are

$$N_{\alpha,\beta}\triangleq x^{-\alpha}Dx^{\beta} \qquad (3.1)$$

$$M_{\alpha,\beta}\triangleq x^{\alpha}Dx^{-\beta} \qquad (3.2)$$

Theorem 3.1

a) The differential operator $N_{\alpha,\beta}\triangleq x^{-\alpha}Dx^{\beta}$ and $\psi\to N_{\alpha,\beta}\psi$ is continuous from ${}^pS_{\beta,a_m,A}$ into ${}^pS_{\beta+1,a_m,A}$ in (2.4).
b) Let the sequence (b_q) satisfy (2.5). Then, the mapping $\psi\to N_{\alpha,\beta}\psi$ is a continuous linear mapping from ${}^pS_\beta^{b_q,B}$ into ${}^pS_{\beta+1}^{b_q,B}$.
c) If the sequence (b_q) satisfy (2.5); $\psi\to N_{\alpha,\beta}\psi$ is a continuous linear mapping of ${}^pS_{\beta,a_m,A}^{b_q,B}$ into ${}^pS_{\beta+1,a_m,A}^{b_q,B}$.

Proof (i) It is straightforward consequence of (2.4) and (3.1)

Proof (ii) Let $\psi\in{}^pS_\beta^{b_q,B}$. By virtue of (2.5) and (3.1);

$$\left|x^m D^q\left(x^\beta\left(N_{\alpha,\beta}\psi(x)\right)\right)\right|\le C_{m,\rho}(B+\rho)^{q+1}b_{q+1}$$

Employing (2□4) yields

$$\left|x^m D^q\left(x^\beta\left(N_{\alpha,\beta}\psi(x)\right)\right)\right|\le C_{m,\rho}(B+\rho)^{q+1}S_1T_1^{q+1}b_q b_1$$

Implies

$$\left|x^m D^q\left(x^\beta\left(N_{\alpha,\beta}\psi(x)\right)\right)\right| \le C'_{m,\rho}\left(BT_1+\rho'\right)^{q+1}$$

where $C'_{m,\rho}=C_{m,\rho}S_1 b_q b_1;\ \rho'=T_1\rho$.

This proves Part (ii).

Proof of (iii). Let $\psi \in {}^pS^{b_q,B}_{\beta,a_m,A}$, then

$$\left|x^m D^q\left(x^\beta\left(N_{\alpha,\beta}\psi(x)\right)\right)\right|$$
$$\le C_{\delta,\rho}(A+\delta)^m(B+\rho)^{q+1}a_m b_{q+1}$$
$$\le C_{\delta,\rho}(A+\delta)^m(B+\rho)^{q+1}a_m S_1 T_1^{q+1} b_q b_1$$
$$\le C'_{\delta,\rho}(A+\delta)^m a_m\left(BT_1+\rho'\right)^{q+1}$$

This completes the proof of the theorem.

In an attempt to provide alternative differential operators which possess linearity among the test function spaces generalized differential operator defined from $N_{\alpha,\beta}$ and $M_{\alpha,\beta}$ is defined by

$$\vartheta_\beta\psi(x)=x^\beta D x^{-\beta}\psi(x) \qquad (3.3)$$

The operator (3.3) is shown to possess the property of linearity of the rapid spaces and hence the corresponding duals as follows [10]:

Theorem 3.2.

a) Let the sequence (a_m) satisfy (2. 3). Then the mapping ${}^pS_{\beta,a_m,A}\to{}^pS_{\beta,a_m,AT}$ and $\psi\to\vartheta_\beta\psi$ is a continuous linear map.

b) Let (b_q) satisfy (2. 4). Then, the map $\psi\to\vartheta_\beta\psi$ is a continuous linear map ${}^pS^{b_q,B}_\beta$ into ${}^pS^{b_q,BT_1}_{\beta+1}$.

c) Let (a_m) and (b_q) satisfy (2.3) and (2.4), respectively. The operation $\psi\to\vartheta_\beta\psi$ maps ${}^pS^{b_q,B}_{\beta,a_m,A}$ into ${}^pS^{b_q,BT_1}_{\beta,a_m,AT}$ continuously.

Proof of (a) Let $\psi\in{}^pS_{\beta,a_m,A}$. With the aid of (3.3),

$$\left|x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)\right|$$
$$=\left|x^m D^q x^\beta\left[x^\beta D x^{-\beta}\psi(x)\right]\right|$$
$$=\left|x^m D^q x^{2\beta} D x^{-\beta}\psi(x)\right|$$
$$=\left|x^m D^{q-1}(2\beta)x^{2\beta-1}Dx^{-\beta}\psi(x)+x^m D^{q-1}x^{2\beta}D^2x^{-\beta}\psi(x)\right|$$
$$=\left|\begin{matrix}x^m D^{q-2}(2\beta)(2\beta-1)x^{2\beta-2}Dx^{-\beta}\psi(x)\\+x^m D^{q-2}(2\beta)x^{2\beta-1}D^2x^{-\beta}\psi(x)+\\x^m D^{q-2}(2\beta)x^{2\beta-1}D^2x^{-\beta}\psi(x)+\\x^m D^{q-2}(2\beta)x^{2\beta-1}D^3x^{-\beta}\psi(x)\end{matrix}\right|$$
$$=\left|\begin{matrix}x^m D^{q-2}(2\beta)(2\beta-1)x^{2\beta-2}Dx^{-\beta}\psi(x)\\+2x^m D^{q-2}(2\beta)x^{2\beta-1}D^2x^{-\beta}\psi(x)+\\x^m D^{q-2}(2\beta)x^{2\beta-1}D^3x^{-\beta}\psi(x)\end{matrix}\right|$$
$$=\left|\begin{matrix}x^m D^{q-3}(2\beta)(2\beta-1)(2\beta-2)x^{2\beta-3}Dx^{-\beta}\psi(x)\\+4x^m D^{q-3}(2\beta)(2\beta-1)x^{2\beta-2}D^3x^{-\beta}\psi(x)\\+x^m D^{q-3}(2\beta)x^{2\beta-1}D^4x^{-\beta}\psi(x)\end{matrix}\right|$$

$$\left|x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)\right|$$
$$\le\left|4x^m D^{q-3}(2\beta)(2\beta-1)x^{2\beta-2}D^3x^{-\beta}\psi(x)\right|$$
$$+\left|x^m D^{q-3}(2\beta)x^{2\beta-1}D^4x^{-\beta}\psi(x)\right|$$
$$=(2q)(-2\beta)\left|x^m x^{2\beta-q}D^q x^{-\beta}\psi(x)\right|$$
$$+\left|x^m x^{2\beta-q+1}D^{q+1}x^{-\beta}\psi(x)\right|$$

This yields

$$x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)$$
$$\le(2q)(-2\beta)C_q^\beta(A+\delta)^m a_m$$
$$+C_q^\beta(A+\delta)^{m+1}a_{m+1}$$
$$\le(2q)(-2\beta)C_q^\beta(A+\delta)^m a_m$$
$$+C_q^\beta(A+\delta)^m(A+\delta)a_m ST^m Ta_1$$

Thus can be rewritten as

$$x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)\le C_q'^\beta(TA+\delta')a_m$$

where $\delta'=\delta T$ and $C_q'^\beta$ is a constant. Thus proved part (a).

Part (b) Let $\psi\in{}^pS^{b_q,B}_\beta$. With the aid of (3.3),

$$\left|x^m D^q x^\beta \left(\vartheta_\beta \psi(x)\right)\right|$$
$$=\left|x^m D^q x^\beta \left[x^\beta D x^{-\beta}\psi(x)\right]\right|$$
$$=\left|x^m D^q x^{2\beta} D x^{-\beta}\psi(x)\right|$$
$$=\left|\begin{matrix} x^m D^{q-1}(2\beta)x^{2\beta-1}Dx^{-\beta}\psi(x) \\ +x^m D^{q-1}x^{2\beta}D^2x^{-\beta}\psi(x)\end{matrix}\right|$$
$$x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)$$
$$\le(2q)(-2\beta)C_{m,\rho}(B+\rho)^q b_q + C_{m,\rho}(B+\rho)^{q+1}b_{q+1}$$
$$\le(2q)(-2\beta)C_{m,\rho}(B+\rho)^q b_q + C_{m,\rho}(B+\rho)^{q+1}S_1T_1^{q+1}b_qb_1$$

Thus can be rewritten as

$$x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)\le C'_{m,\rho}(BT_1+\rho')b_q$$

where $C'_{m,\rho}=C_{m,\rho}S_1b_qb_1$; $\rho'=T_1\rho$ is a constant.

Thus proved part (b).

Part (c) Let $\psi\in {}^pS^{b_q,B}_{\beta,a_m,A}$.

$$x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)$$
$$\le(2q)(-2\beta)C_q^\beta C_{m,\rho}(A+\delta)^m(B+\rho)^q a_m b_q + C_q^\beta C_{m,\rho}(A+\delta)^{m+1}(B+\rho)^{q+1}a_{m+1}b_{q+1}$$

$$\le(2q)(-2\beta)C_q^\beta C_{m,\rho}(A+\delta)^m(B+\rho)^q b_q a_m + C_q^\beta(A+\delta)^m(A+\delta)(B+\rho)^{q+1}S_1T_1^{q+1}b_qb_1a_mST^mTa_1.$$

Thus it yields

$$x^m D^q x^\beta\left(\vartheta_\beta\psi(x)\right)\le C'_{m,\rho}C'^\beta_q(BT_1+\rho')(TA+\delta')b_qa_m$$

where $C'_{m,\rho}C'^\beta_q=C_{m,\rho}S_1b_qb_1a_mST^mTa_1$; $\rho'=T_1\rho$; $\delta'=\delta T$ is a constant. Thus proved part (c).

4 GENERALIZED HANKEL-CLIFFORD TRANSFORMATION OF TEMPERED ULTRADISTRIBUTIONS

The generalized Hankel Clifford transformation the duals of Roumieu type tempered ultradistributions is defined as the adjoint operator.

$$D_x\left(x^{-\beta}\mathrm{J}_{\alpha,\beta}(xy)\right)=x^{-\beta}\mathrm{J}_{\alpha-\beta+1}(xy). \qquad (4.1)$$

Theorem 4.1 Let α,β be real numbers, $\alpha-\beta\ge-1/2$. Let the sequence (a_m) satisfy (2.3) of [5]. The transform is an automorphism from ${}^pS_{\beta,a_m,A}\to{}^pS_{\beta,a_m,AT}$.

Proof: Using (4.1) leads to

$$\left|y^m D_y^q y^\beta\left(h_{\alpha,\beta}\psi\right)(x)\right|$$
$$=\left|(-1)^q y^{\alpha+m}\int_0^\infty x^{2q}\mathrm{J}_{\alpha+q,\beta+q}(xy)\phi(x)dx\right|$$
$$=\left|(-1)^q y^{\alpha+m}\int_0^\infty\left(x^{-\beta+2q}\mathrm{J}_{\alpha+q,\beta+q}(xy)\right)x^\beta\phi(x)dx\right|$$

$$\left|y^m D_y^q y^\beta\left(h_{\alpha,\beta}\psi\right)(x)\right|$$
$$\le C_{m,q,\alpha,\beta}\left|\int_0^\infty x^pD^mx^\beta\phi(x)dx\right|$$
$$\le C_{m,q,\alpha,\beta}\sum_{k=0}^{p+2}x^k D^m x^\beta\phi(x)$$
$$\le C_{m,q,\alpha,\beta}\sum_{k=0}^{p+2}C_q^\beta(A+\delta)^k a_k$$
$$\le C'_{m,q,\alpha,\beta}(A+\delta)^{p+2}\sum_{k=0}^{p+2}a_k$$
$$\left|y^m D_y^q y^\beta\left(h_{\alpha,\beta}\psi\right)(x)\right|$$
$$\le C'_{m,q,\alpha,\beta}(p+3)(A+\delta)^2ST^2a_2\left((A+\delta)T\right)^p a_p$$

for some constant $C'_{m,q,\alpha,\beta}$. Implies $h_{\alpha,\beta}\in{}^pS_{\beta,a_m,AT}$.

Thus the proof.

Theorem 4.2 Let α,β be real numbers, $\alpha-\beta\ge-1/2$. Let the sequence (a_m) and (b_q) satisfy (2.3) of [5]. The transform is an automorphism as in [9] from ${}^pS^{b_q,B}_{\beta,a_m,A}\to{}^pS^{b_q,BT_1}_{\beta,a_m,AT}$.

Proof: Using (4.1) leads to

$$\left|y^m D_y^q y^\beta \left(h_{\alpha,\beta}\psi\right)(x)\right| \le C_{m,q,\alpha,\beta} \sum_{k=0}^{p+2} C_q^\beta \left(A+\delta\right)^k \left(B+\rho\right)^q b_q a_k$$

$$\left|y^m D_y^q y^\beta \left(h_{\alpha,\beta}\psi\right)(x)\right| \le C_{m,q,\alpha,\beta} C_q^\beta \left(p+3\right)\left(A+\delta\right)^2 \times ST^2 a_2 \left(\left(A+\delta\right)T\right)^p \left(B+\rho\right)^q b_q a_p .$$

$$\left|y^m D_y^q y^\beta \left(h_{\alpha,\beta}\psi\right)(x)\right| \le \varpi_{m,q,\alpha,\beta} \left(A+\delta'\right)^p \left(B+\rho\right)^q b_q a_p$$

where

$$\varpi_{m,q,\alpha,\beta} = C_{m,q,\alpha,\beta} C_q^\beta \left(p+3\right)\left(A+\delta\right)^2 ST^2 a_2 .$$

Thus the proof.

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