Evaluation of Integrals Containing Bessel's Functions Using Kushare Transform

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Abstract: Integral transform plays very important role in various fields. In this paper we use Kushare transform for Bessel's function. Further we use it to solve integrals.
Key Words: Bessel's functions, Kushare transform Integral transform.

1. INTRODUCTION:

Recently, integral transforms are one of the most useful and simple mathematical technique for obtaining the solutions of advance problems occurred in many fields like science, Engineering, technology, commerce and economics. To provide exact solution of problem without lengthy calculations is the important feature of integral transforms.

Due to this important feature of the integral transforms many researchers are attracted to this field and are engaged in introducing various integral transforms. Recently, Kushare and Patil [1] introduced new integral transform called as Kushare transform for solving differential equations in time domain. Further, Savita Khakale and Dinkar Patil [2] introduced Soham transform in November 2021. As researchers are interested in introducing the new integral transforms at the same time they are also interested in applying the transforms to various fields, various equations in different domain. In January 2022, Sanap and Patil [3] used Kushare transform for obtaining the solution of the problems on Newton's law of Cooling.

In April 2022 D. P. Patil, et al [4] solved the problems on growth and decay by using Kushare transform. D.P. Patil [5] also used Sawi transform in Bessel functions. Further, Patil [6] evaluate improper integrals by using Sawi transform of error functions. Laplace transforms and Shehu transforms are used in chemical sciences by Patil [7]. Dinkar Patil [8] used Sawi transform and its convolution theorem for solving wave equation. Using Mahgoub transform, parabolic boundary value problems are solved by D .P. Patil [9].

D .P. Patil [10] used double Laplace and double Sumudu transforms to obtain the solution of wave equation. Further Dr. Patil [11] also obtained dualities between double integral transforms. Kandalkar, Gatkal and Patil [12] solved the system of differential equations using Kushare transform. D. P. Patil [13] solved boundary value problems of the system of ordinary differential equations by using Aboodh and Mahgoub transforms. Double Mahgoub transformed is used by Patil [14] to solve parabolic boundary value problems. Laplac, Sumudu , Aboodh , Elazki and Mahagoub transforms are compared and used it for solving boundary value problems by Dinkar Patil [15]. D. P. Patil et al [16] solved Volterra Integral equations of first kind by using Emad-Sara transform. Futher Derle, Rahane and Patil[17] introduced general double rangaig integral transform. Rathi sisters and D. P. Patil [18] solved system of differential equations by using Soham transform. Vispute, Jadhav and Patil [19] used Emad Sara transform for solving telegraph equation. Kandalkar, Zankar and Patil [20] evaluate the improper integrals by using general integral transform of error function. Dinkar Patil, Prerana Thakare and Prajakta Patil[21] obtained the solution of parabolic boundary value problems by using double general integral transform. Dinkar Patil used Emad- Falih transform for solving problems based on Newton's law of cooling [22]. D. P. Patil et al [23] used Soham transform to obtain the solution of Newton's law of cooling. Dinkar Patil et al [24] used HY integral transform for handling growth and Decay problems, D. P. Patil et al used HY transform for Newton's law of cooling [25]. D. P. Patil et al [26] used Emad-Falih transform for general solution of telegraph equation. Dinkar Patil et al [27] introduced double kushare transform. Recently, D. P. Patil et al [28] solved population growth and decay problems by using Emad Sara transform. Alenzi transform is used in population growth and decay problems by patil et al [29]. Thete, et al [30] used Emad Falih transform for handling growth and decay problems. Nikam, Patil et al [31] used , Kushare transform of error functions in evaluating improper integrals. Wagh sisters and Patil used Kushare [32] and Soham [33] transform in chemical Sciences. Malpani, Shinde and Patil [34] used Convolution theorem for Kushare transform and applications in convolution type Volterra integral equations of first kind. Raundal and Patil [35] used double general integral transform for solving boundary value problems in partial differential equations. Rahane, Derle and Patil [36] eveloped generalized double rangaig integral transform. Kushare transform is used for solving Volterra Integro-Differential equations of first kind by Shinde, et al [37]. Kandekar et al [38] used new general integral equation to solve Abel's integral equations. Pardeshi, Shaikh and Patil[39] used Kharrat Toma transform for solving population growth and decay problems.

2. PRELIMINARIES :

In this section we state some preliminary concepts which are required.

Bessel's functions have many important applications in solving problems of mathematical physics, engineering, acoustics and sciences such as fluid mechanics, hydrodynamics, heat transfer, vibrations, stress analysis, flux distribution in a nuclear reactor, problem of pharmacokinetics, and nuclear physics etc.

Bessel's function of order n, [5] where n is the natural number is given by

$$J_n(t) = \frac{t^n}{2^n n!}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \frac{t^6}{2.4.6(2n+2)(2n+4)(2n+6)} + \cdots\right] \quad \ldots(1)$$

If $n = 0$, Bessel's function of order zero is denoted by $J_0(t)$ and becomes

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots \quad \ldots(2)$$

For $n = 1$, we get Bessel's function of order one [5] and is denoted by $J_1(t)$

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!.3!} - \frac{t^7}{2^7.3!.4!} + \cdots \ldots(3)$$

For $n = 2$, we get Bessel's function of order two and denoted by $J_2(t)$

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2 4^2.6.8} - \frac{t^7}{2^7.4^2.6^2.8.10} + \cdots \quad \ldots(4)$$

If $t = 0$ then $J_0(0) = 1$ and $J_1(0) = J_2(0) = 0$

2.1. Kushare transform Definition: The Kushare integral transform of the function $f(t)$ is defined,[1]:

$$K[f(t)] = s(v) = v\int_0^\infty f(t)e^{-tv^{\propto}}dt\ ,\ t \geq 0, T_1 \leq v \leq T_2 \quad ..(5)$$

The Kushare transform of the function $f(x)$ exist if $f(t)$ is a piecewise continuous and of exponential order.

These are sufficient conditions for the existence of Kushare integral transform of the function $f(t)$.

2.2. Kushare Transform and Inversion of the some preliminary functions:

$K(1) = \frac{1}{v^{\propto-1}} = 1 = S(v)$ Inversion formula $K^{-1}\left(\frac{1}{v^{\propto-1}}\right) = 1 = f(t)$

$K(t^n) = \frac{\Gamma(n+1)}{v^{\propto(n+1)}} = S(v)$ Inversion formula: $K^{-1}\left(\frac{\Gamma(n+1)}{v^{\propto(n+1)}}\right) = t^n = f(t)$

$K(e^{at}) = \frac{v}{v^{\propto}-a} = S(v)$ Inversion formula $K^{-1}\left(\frac{v}{v^{\propto}-a}\right) = e^{at} = f(t)$

$K(\sin at) = \frac{av}{v^{2\propto}+a^2} = S(v)$ Inversion formula: $K^{-1}\left(\frac{v}{v^{2\propto}+a^2}\right) = \frac{\sin at}{a} = f(t)$

$K(\cos at) = \frac{v^{\propto+1}}{v^{2\propto}+a^2} = S(v)$ Inversion formula: $K^{-1}\left(\frac{v^{\propto+1}}{v^{2\propto}+a^2}\right) = \cos at = f(t)$

2.3. Kushare Transform of derivatives:

If, $K[f(t)] = S(v)$then,

1.$K[f'(t)] = v^{\propto}S(v) - vf(0)$

2.$K[f''(t)] = v^{2\propto}S(v) - v^{\propto+1}f(0) - vf'(0)$

3.K$[K[f^n(t)] = S(v) - v\sum_{k=0}^{n-1} v^{\propto(n-k-1)}f^k(0)$

2.4. Properties of Kushare Transform:

In this section we state and prove some properties of Kushare transform

2.4.1. Change of scale property:

If Kushare transform of function $F(t)$is $S(v)$then Kushare function $F(at)$ is given by

$K[f(at)] = a^{\propto^{-1}-1}S\left(\frac{v}{\sqrt[\propto]{a}}\right)$.

Proof:

We know by definition of Kushare transformation

$$K[f(t)] = v\int_0^\infty f(t)e^{-tv^{\propto}}dt$$

Now, $K[f(at)] = v\int_0^\infty f(at)e^{-tv^{\propto}}dt$

Put, $at = p \Rightarrow t = \frac{a}{p}$ $\Rightarrow adt = dp$

$$K[f(at)] = v\int_0^\infty \frac{1}{a}f(p)e^{-\frac{-p}{a}v^{\propto}}dt$$

$$K[f(at)] = \frac{v}{a}\int_0^\infty f(p)e^{-p\left(\frac{v}{\sqrt[\propto]{a}}\right)^{\propto}}dt$$

$$K[f(at)] = \frac{v}{a}\frac{\sqrt[\propto]{a}}{\sqrt[\propto]{a}}\int_0^\infty f(p)e^{-p\left(\frac{v}{\sqrt[\propto]{a}}\right)^{\propto}}dt$$

$$K[f(at)] = \frac{a^{1/\propto}}{a}\frac{v}{\sqrt[\propto]{a}}\int_0^\infty f(p)e^{-p\left(\frac{v}{\sqrt[\propto]{a}}\right)^{\propto}}dt$$

$$K[f(at)] = a^{\propto^{-1}-1}s\left(\frac{v}{\sqrt[\propto]{a}}\right)$$

Thus the property is proved.

2.2.2. Shifting property:

If, Kushare transform of function $f(t)$is $s(v)$then Kushare Transform of $e^{at}f(t)$ is given by $K[e^{at}f(t)] = \frac{v}{(v^{\propto}-a)^{1/\propto}}s\left(\sqrt[\propto]{v^{\propto}-a}\right)$.

Proof: We know by the definition of Kushare transformation ,

$$K[f(t)] = v\int_0^\infty f(t)e^{-tv^{\propto}}dt$$

Now, $K[e^{at}f(t)] = v\int_0^\infty e^{at}f(t)e^{-tv^{\propto}}dt$

$$K[e^{at}f(t)] = v\int_0^\infty f(t)e^{at-tv^{\propto}}dt$$

$$K[e^{at}f(t)] = v\int_0^\infty f(t)e^{-t(v^{\propto}-a)}dt$$

$$K[e^{at}f(t)] = v\int_0^\infty f(t)e^{-t\left(\sqrt[\propto]{v^{\propto}-a}\right)^{\propto}}dt$$

$$K[e^{at}f(t)] = \frac{v}{\sqrt[\propto]{v^{\propto}-a}}\sqrt[\propto]{v^{\propto}-a}\int_0^\infty f(t)e^{-t\left(\sqrt[\propto]{v^{\propto}-a}\right)^{\propto}}dt$$

$$K[e^{at}f(t)] = \frac{v}{(v^{\propto}-a)^{1/\propto}}s\left(\sqrt[\propto]{v^{\propto}-a}\right)$$

Thus the property is proved.

2.2.3. Kushare Transform of function $tf(t)$:

If,$K[f(t)] = s(v)$, then $K[tf(t)] = \left(\frac{1}{\propto v^{\propto-1}}\right)\left[\frac{d}{dv} - \frac{1}{v}\right]S(v)$.

Proof: Let, $K[f(t)] = S(v)$

We know by Kushare transformation definition, $K[f(t)] = v\int_0^\infty f(t)e^{-tv^\propto}dt$

Now differentiate this equation with respect to v, we get

$$\frac{d}{dv}S(v) = \frac{d}{dv}\left(v\int_0^\infty f(t)\,e^{-tv^\propto}dt\right)$$

$$= \int_0^\infty f(t)\frac{d}{dv}ve^{-tv^\propto}dt$$

$$= \int_0^\infty f(t)\left[ve^{-tv^\propto}(-t\propto v^{\propto-1}) + e^{-tv^\propto}\right]dt$$

$$= \int_0^\infty f(t)ve^{-tv^\propto}(-t\propto v^\propto)dt + \int_0^\infty f(t)e^{-tv^\propto}\,dt$$

$$= \frac{1}{v}\left[v\int_0^\infty f(t)ve^{-tv^\propto}(-t\propto v^{-tv^\propto})dt\right] + \frac{1}{v}\int_0^\infty f(t)e^{-tv^\propto}dt$$

$$= -\propto v^{\propto-1}\int_0^\infty f(t)te^{-tv^\propto}dt + \frac{1}{v}S(v)$$

$$K[tf(t)] = \left(\frac{1}{\propto v^{\propto-1}}\right)\left[\frac{d}{dv}S(v) - \frac{1}{v}S(v)\right]$$

$$K[tf(t)] = \left(\frac{1}{\propto v^{\propto-1}}\right)\left[\frac{d}{dv} - \frac{1}{v}\right]S(v)$$

Thus the property is proved.

2.2.4 Convolution theorem for Kushare Transform:

If Kushare transform of function $f(t)$ and $g(t)$ are $f(v)$and $g(v)$ respectively then Kushare transform of their convolution $f(t)\&g(t)$ is given by,

$$K[f(t)*g(t)] = \frac{1}{v}f(v).g(v)$$

Proof: The convolution of function $f(t)$ and $g(t)$ is $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

Applying Kushare transformation, we get $K[(f*g)(t)] = K\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]$

We now by definition of Kushare Transformation

$$K[f(t)] = v\int_0^\infty f(t)e^{-tv^\propto}dt$$

$$K[(f*g)(t)] = v\int_0^\infty e^{-tv^\propto}\int_0^t f(\tau)g(t-\tau)d\tau\,dt$$

$$= v\int_0^\infty \int_\tau^\infty e^{-tv^\propto}f(\tau)g(t-\tau)d\tau\,dt$$

Now, set $-\tau = b$, we get

$$K[(f*g)(t)] = v\int_0^\infty \int_0^\infty e^{-(\tau+b)v^\propto}f(\tau)g(b)dbd\tau$$

$$= v\int_0^\infty \int_0^\infty e^{-\tau v^\propto}e^{-bv^\propto}f(\tau)g(b)dbd\tau$$

$$= \frac{1}{v}\left\{v\int_0^\infty e^{-tv^\propto}f(\tau)d\tau.\,v\int_0^\infty e^{-bv^\propto}g(b)db\right\}$$

$$K[f(t)*g(t)] = \frac{1}{v}f(v).g(v)$$

Thus the convolution theorem is proved.

2.2.5. Relation Between $\boldsymbol{J_0(t)}$ and $\boldsymbol{J_1(t)}$:

$$\frac{d}{dt}J_0(t) = -J_1(t)$$

Relation Between $\boldsymbol{J_0(t)}$ and $\boldsymbol{J_2(t)}$:

$$J_2(t) = J_0(t) + 2J''_0(t)$$

3. KUSHARE TRANSFORM OF BESSEL'S FUNCTION: Bessel's function is defined as

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots$$

Applying Kushare Transform to the both sides, we get

$$K[J_0(t)] = K\left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots\right]$$

$$= K(1) - \frac{1}{2^2}K(t^2) + \frac{1}{2^2.4^2}K(t^4) - \frac{1}{2^2.4^2.6^2}K(t^6)$$

$$= \frac{1}{v^{\propto-1}} - \frac{1}{2^2}\frac{\Gamma 3}{v^{3\propto-1}} + \frac{1}{2^2.4^2}\frac{\Gamma 5}{v^{5\propto-1}} - \frac{1}{2^2.4^2.6^2}\frac{\Gamma 7}{v^{7\propto-1}} + \cdots$$

$$= \frac{1}{v^{\propto-1}}\left[1 - \frac{1}{2^2}\frac{2!}{v^{2\propto}} + \frac{1}{2^2.4^2}\frac{4!}{v^{4\propto}} - \frac{1}{2^2.4^2.6^2}\frac{6!}{v^{6\propto}} + \cdots\right]$$

$$= \frac{1}{v^{\propto-1}}\left[1 - \frac{1}{2^2}\frac{2}{v^{2\propto}} + \frac{1}{2^2.4^2}\frac{24}{v^{4\propto}} - \frac{1}{2^2.4^2.6^2}\frac{720}{v^{6\propto}} + \cdots\right]$$

$$= \frac{1}{v^{\propto-1}}\left[1 - \frac{1}{2v^{2\propto}} - \frac{3}{8}\frac{1}{v^{4\propto}} - \frac{5}{16}\frac{1}{v^{6\propto}} + \cdots\right]$$

$$= \frac{1}{v^{\propto-1}}\left[\frac{1}{\sqrt{1+\frac{1}{v^{2\propto}}}}\right]$$

$$K[J_0(t)] = \frac{v}{\sqrt{v^{2\propto}+1}}$$

$$K[J_1(t)] = -K\left[\frac{d}{dt}J_0(t)\right]$$

$$= -[v^\propto K[J_0(t)] - vJ(0)]$$

$$= -\left[v^\propto\left[\frac{v}{\sqrt{v^{2\propto}+1}}\right] - v\right]$$

$$= -\left[\frac{v^{\propto+1}}{\sqrt{v^{2\propto}+1}} - v\right]$$

$$K[J_1(t)] = -\frac{v^{\propto+1}}{\sqrt{v^{2\propto}+1}} + v$$

$$K[J_2(t)] = K[J_0(t)] + 2K[J_0''(t)]$$

$$= \frac{v}{\sqrt{v^{2\propto}+1}} + 2\left[v^{2\propto}\left[\frac{v}{\sqrt{v^{2\propto}+1}}\right] - v^{\propto+1} - v(0)\right] \ldots (J(0) = 1 \,\&J'(0) = 0)$$

$$= \frac{v}{\sqrt{v^{2\propto}+1}} + 2\left[\frac{v^{2\propto+1}}{\sqrt{v^{2\propto}+1}} - v^{\propto+1}\right]$$

$$K[J_0(at)] = \frac{v}{\sqrt{v^{2\propto}+1}}$$

By using Change of scale property, we get

$$K[J_0(at)] = a^{\left(\frac{1}{\propto}-1\right)}\frac{v}{\sqrt{\frac{v^{2\propto}}{a^{2/\propto}}+1}} = a^{1/\propto}a^{-1}\frac{v}{\sqrt{\frac{v^{2\propto}+a^{2/\propto}}{a^{2/\propto}}}}$$

$$= a^{1/\propto}a^{-1}\left(\frac{va^{1/\propto}}{\sqrt{v^{2\propto}+a^{2/\propto}}}\right) = \frac{va^{2/\propto}}{a\sqrt{v^{2\propto}+a^{2/\propto}}}$$

and

$$K[J_1(at)] = -\frac{v^{\propto+1}}{\sqrt{v^{2\propto}+1}} + v^\propto = -v^\propto\left[\frac{v}{\sqrt{v^{2\propto}+1}} - 1\right] = -a^{\left(\frac{1}{\propto}-1\right)}v^\propto\left[\frac{v}{\sqrt{v^{2\propto}+1}} - 1\right] = -\left[\frac{v^{\propto+1}a^{1/\propto}}{\sqrt{\frac{v^{2\propto}+a^{2/\propto}}{a^{2/\propto}}}} - \frac{v^\propto a^{1/\propto}}{a.a^{1/\propto}}\right]$$

$$= -\left[\frac{v^{\propto+1}a^{2/\propto}}{a\sqrt{v^{2\propto}+a^{2/\propto}}} - \frac{v^{\propto}}{a}\right] = -\frac{v^{\propto}}{a}\left[\frac{v.a^{2/\propto}}{\sqrt{v^{2\propto}+a^{2/\propto}}} - 1\right]$$

4. APPLICATIONS:

In this section, some applications are given in order to demonstrate the effectiveness of Kushare Transform of Bessel's functions for calculating the integral which contain Bessel's functions.

Application 1: Calculate the integral $I(t) = \int_0^t J_0(t)J_0(t-u)du$

Solution: Applying Kushare transform to both sides of given equation, we get

$$K[I(t)] = K\left[\int_0^t J_0(t)J_0(t-u)du\right]$$

By convolution theorem, $K[I(t)] = \frac{1}{v}K[J_0(t)]K[J_0(t)]$

$$= \frac{1}{v}\left[\frac{v}{\sqrt{v^{2\propto}+1}}\right]\left[\frac{v}{\sqrt{v^{2\propto}+1}}\right] = \frac{v}{v^{2\propto}+1}$$

Taking inverse of both sides, we get

$I(t) = \sin t$ …(by inversion formula for Kushare Transform)

Application 2: Evaluate the integral, $I(t) = \int_0^t J_0(t)J_1(t-u)du$

Solution: Applying Kushare transform to both sides of given equation, we get

$$K[I(t)] = K\left[\int_0^t J_0(t)J_1(t-u)du\right]$$

By convolution theorem, $K[I(t)] = \frac{1}{v}K[J_0(t)]K[J_1(t)]$

$$= \frac{1}{v}\left[\frac{v}{\sqrt{v^{2\propto}+1}}\right]\left[-\frac{v^{\propto+1}}{\sqrt{v^{2\propto}+1}} + v\right]$$

$$= -\frac{v^{\propto+1}}{v^{2\propto}+1} + \frac{v}{\sqrt{v^{2\propto}+1}}$$

Taking inverse of both sides, we get

$I(\mathrm{t}) = -\cos t + J_0(t)$…(by inversion formula for Kushare Transform)

$$I(\mathrm{t}) = J_0(t) - \cos t$$

Application 3: Evaluate the integral, $I(t) = \int_0^t J_1(t-u)du$

Solution: Applying Kushare transform to both sides of given equation, we get

$$K[I(t)] = K\left[\int_0^t J_1(t-u)du\right]$$

By convolution theorem, $K[I(t)] = \frac{1}{v}K[1].K[J_1(t)]$

$$= \frac{1}{v}.\frac{1}{v^{\propto-1}}\left[-\frac{v^{\propto+1}}{\sqrt{v^{2\propto}+1}} + v\right]$$

$$= -\frac{v^{\propto+1}}{v.v^{\propto-1}\sqrt{v^{2\propto}+1}} + \frac{v}{v.v^{\propto-1}}$$

$$= -\frac{v}{\sqrt{v^{2\propto}+1}} + 1$$

Taking inverse of both sides, we get $I(t) = 1 - J_0(t)$

5. Conclusion: In this paper, we discussed the Kushare Transform of Bessel's functions. Also, the given applications show that the advantage of Kushare transform of Bessel's Functions to calculate the integral which contain Bessel's functions.

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