Abstract: Present paper aims at investigating a theory supporting generalized Hankel-Clifford transformation on certain spaces of generalized functions. In an attempt to extend the transform to a space of tempered ultradistributions; definitions of classes of rapid descent ultradifferentiable functions is provided. Mappings involving various differential operators are shown to be continuous.

Keywords: Generalized Hankel-Clifford transformation, tempered ultradifferentiable functions, differential operator

1 INTRODUCTION

In [1] the author studied Bessel Functions and their Applications to Physics. In [2] Ultradistributions: Structure Theorems and a Characterization have been studied. Integral transforms of generalized functions and their applications has been studied in [3]. In [6, 7] the author observed the Hankel-Clifford transformation on certain spaces of ultradistributions. In 2009, [5] author has analyzed generalized Hankel-type transformation for a class of tempered ultradistributions of Roumieu-type.

Malgonde [4] investigated the variant of the generalized Hankel-Clifford transform defined by

\[(h_{a,\beta}f)(y) = \mathcal{F}(y) = \int_{0}^{\infty} (y/x)^{(a+\beta)/2} J_{a,\beta}(2\sqrt{xy}) f(x) dx, \quad (\alpha - \beta) \geq -1/2 \]

\[(h_{a,\beta}f)(y) = y^{-a-\beta} \int_{0}^{\infty} J_{a,\beta}(xy) f(x) dx, \quad (\alpha - \beta) \geq -1/2 \]  

(1.1)

where \( J_{a,\beta}(x) = (x)_{a,\beta} J_{a,\beta}(2\sqrt{x}), \) \( J_{a,\beta}(x) \) being the Bessel function of the first kind of order \( (\alpha - \beta) \), in spaces of generalized functions.

Certain results for certain differential operators and further, make use of a new defined differential operator and accordingly prove new relevant theorem are established. Hankel-type transform to spaces of tempered ultradistributions of Roumieu type have recently been discussed in [5].

2 ULTRADIFFERENTIABLE FUNCTIONS

The notation and terminology used in [4] will be continued. Sequences \( a_m \) and \( b_q, m, q = 1, 2, 3, \ldots \) are sequences of positive real numbers imposed by some of the following constraints as in [5].

Definition 2.1. Let \( a > 0 \) an arbitrary constant, \( \alpha - \beta \in R \) and \( p \in N \). Define the function of space \( \mathcal{S}_{\alpha,\beta}^{a,A} \) as the collection of all complex valued smooth functions \( \psi \) defined on \( I(0,\infty) \) such that set of all infinitely smooth functions satisfying

\[ \left| x^m D^q \left( x^\beta \psi(x) \right) \right| \leq C_{\alpha,\beta} \left( A + \delta \right)^m a_m \]

for every \( m, q \in N \) and \( \delta > 0 \). Let \( B > 0, b_q \geq 0 \) an arbitrary constant \( \alpha - \beta \in R \).

Define the function of space \( \mathcal{S}_{\alpha,\beta}^{b,A} \) as the collection of all complex valued smooth functions \( \psi(x) \) defined on \( I(0,\infty) \) such that set of all infinitely smooth functions satisfying

\[ \sup_{x \in I} \left| x^m D^q \left( x^\beta \psi(x) \right) \right| \leq C_{\alpha,\beta} \left( B + \rho \right)^q b_q \]

for every \( m, q \in N \) and \( \rho > 0 \). \( C_{\alpha,\beta} \) are positive constant depending on \( \psi \).

Let \( A, B > 0 \) and \( a, b \geq 0 \) an arbitrary constant \( \alpha - \beta \in R \). Define the function of space \( \mathcal{S}_{\alpha,\beta}^{A,B} \) as the collection of all complex-valued smooth functions \( \psi(x) \) defined on \( I(0,\infty) \) such that set of all infinitely smooth functions satisfying

\[ \sup_{x \in I} \left| x^m D^q \left( x^\beta \psi(x) \right) \right| \leq C_{\alpha,\beta} \left( A + \delta \right)^m \left( B + \rho \right)^q a_m b_q \]

for every \( m, q \in N \) and \( \delta, \rho > 0 \). \( C_{\alpha,\beta} \) are positive constant depending on \( \psi \).

In view of Definition 2.1 \( \mathcal{S}_{\alpha,\beta}^{A,B} \) is a linear space with the usual operations. Moreover, if
\[ \| \psi \|_{\alpha, \delta} = \sup_{x \in D} \frac{|x^m D^q (x^\beta \psi (x))|}{(A + \delta)^m (B + \rho)^q} a_m b_q \]  

(2.4)

for every \( m, q, \delta, \rho > 0 \), each \( \| \psi \|_{\alpha, \delta} \) is a seminorm on \( pS_{\beta,\alpha,A} \) and the collection

\[ \Gamma = \{ \| \psi \|_{\alpha, \delta} \}_{\alpha, \delta > 0} \]  

is a multinorm because each \( \| \psi \|_{\alpha, \delta} \) is a norm. Since the systems of seminorms \( \Gamma \) and \( \Gamma_1 = \{ \| \psi \|_{\alpha, 1/n} \}_{\alpha, n \in N} \) are equivalent, the space \( pS_{\beta,\alpha,A} \) equipped with the topology generated by \( \Gamma_1 \), is a countable multinormed space.

\( pS_{\beta,\alpha,A}^{b,B} \) is a linear space with the usual operations. Moreover, if

\[ \| \psi \|_{\alpha, \delta}^{\alpha, \rho} = \sup_{x \in D} \frac{|x^m D^q (x^\beta \psi (x))|}{(B + \rho)^q} b_q \]  

(2.5)

for every \( q \in N \) and \( \rho > 0 \), each \( \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \) is a seminorm on \( pS_{\beta,\alpha,A}^{b,B} \) and the collection

\[ \Gamma = \{ \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \}_{\alpha, \delta > 0} \]  

is a multinorm because each \( \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \) is a norm in [8]. Since the systems of seminorms \( \Gamma \) and \( \Gamma_1 = \{ \| \psi \|_{\alpha, 1/n} \}_{\alpha, n \in N} \) are equivalent, the space \( pS_{\beta,\alpha,A}^{b,B} \) equipped with the topology generated by \( \Gamma_1 \), is a countable multinormed space.

\( pS_{\beta,\alpha,A}^{b,B} \) is a linear space with the usual operations. Moreover, if

\[ \| \psi \|_{\alpha, \delta}^{\alpha, \rho} = \sup_{x \in D} \frac{|x^m D^q (x^\beta \psi (x))|}{(A + \delta)^m (B + \rho)^q} a_m b_q \]  

(2.6)

for every \( m, q, \delta, \rho > 0 \), each \( \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \) is a seminorm on \( pS_{\beta,\alpha,A}^{b,B} \) and the collection

\[ \Gamma_2 = \{ \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \}_{\alpha, \delta > 0} \]  

is a multinorm because each \( \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \) is a norm. Since the systems of seminorms \( \Gamma \) and \( \Gamma_2 = \{ \| \psi \|_{\alpha, \delta}^{\alpha, \rho} \}_{\alpha, \delta > 0} \) are equivalent, the space \( pS_{\beta,\alpha,A}^{b,B} \) equipped with the topology generated by \( \Gamma_2 \), is a countable multinormed space.

Inspired in the work of Roumieu [5], the dual spaces of \( pS_{\beta,\alpha,A}^{b,B}, pS_{\beta,\alpha,A}^{b,B}, pS_{\beta,\alpha,A}^{b,B} \) are called tempered ultradistributions of Roumieu-type. The generalized Hankel-Clifford transformation transform of the classes on the duals is called tempered ultradistributional generalized Hankel-Clifford transformation of Roumieu-type.

### 3 OPERATIONS OF \( pS_{\beta,\alpha,A}^{b,B}, pS_{\beta,\alpha,A}^{b,B} \) and \( pS_{\beta,\alpha,A}^{b,B} \)

The differential operators are

\[ N_{\alpha, \beta} \Delta x^{-\alpha} D x^{\beta} \]  

(3.1)

\[ M_{\alpha, \beta} \Delta x^{\alpha} D x^{-\beta} \]  

(3.2)

Theorem 3.1

a) The differential operator \( N_{\alpha, \beta} \Delta x^{-\alpha} D x^{\beta} \) and \( \psi \to N_{\alpha, \beta} \psi \) is continuous from \( pS_{\beta,\alpha,A}^{b,B} \) into \( pS_{\beta+1,\alpha,A}^{b,B} \) in (2.4).

b) Let the sequence \( \{ b_q \} \) satisfy (2.5). Then, the mapping \( \psi \to N_{\alpha, \beta} \psi \) is a continuous linear mapping from \( pS_{\beta}^{b,B} \) into \( pS_{\beta+1}^{b,B} \).

c) If the sequence \( \{ b_q \} \) satisfy (2.5); \( \psi \to N_{\alpha, \beta} \psi \) is a continuous linear mapping of \( pS_{\beta,\alpha,A}^{b,B} \) into \( pS_{\beta+1,\alpha,A}^{b,B} \).

Proof (i) It is straightforward consequence of (2.4) and (3.1)

Proof (ii) Let \( \psi \in pS_{\beta}^{b,B} \). By virtue of (2.5) and (3.1);

\[ x^m D^q \left( x^\beta \left( N_{\alpha, \beta} \psi (x) \right) \right) \leq C_{\alpha, \beta} (B + \rho)^{q+1} b_q \]

Employing (2 \( \square \) 4) yields

\[ x^m D^q \left( x^\beta \left( N_{\alpha, \beta} \psi (x) \right) \right) \leq C_{\alpha, \beta} (B + \rho)^{q+1} S_t T_{q+1} b_q b_q \]

Implies
\[ x^\alpha D^\beta \left( N_{\alpha, \beta} \psi \left( x \right) \right) \leq C_{\mu, \rho} \left( BT_i + \rho \right)^{q+1} \]

where \( C_{\mu, \rho} = C_{\mu, \rho} S_{\mu} b_{\mu} \); \( \rho' = T_i \rho \).

This proves Part (ii).

Proof of (iii). Let \( \psi \in S_{\beta, \alpha, A} \), then

\[
\left| x^\alpha D^\beta \left( N_{\alpha, \beta} \psi \left( x \right) \right) \right| \leq C_{\delta, \rho} \left( A + \delta \right)^{m} \left( B + \rho \right)^{q+1} a_{m} b_{q+1} \\
\leq C_{\delta, \rho} \left( A + \delta \right)^{m} \left( B + \rho \right)^{q+1} a_{m} S_{T_i} b_{q+1} b_i \\
\leq C_{\delta, \rho} \left( A + \delta \right)^{m} a_{m} \left( BT_i + \rho \right)^{q+1}
\]

This completes the proof of the theorem.

In an attempt to provide alternative differential operators which possess linearity among the test function spaces generalized differential operator defined from \( N_{\alpha, \beta} \) and \( M_{\alpha, \beta} \) is defined by

\[ \mathcal{D}_{\beta} \psi \left( x \right) = x^\beta D_x \psi \left( x \right) \]  

(3.3)

The operator (3.3) is shown to possess the property of linearity of the rapid spaces and hence the corresponding duals as follows [10]:

Theorem 3.2.

a) Let the sequence \( \left( a_m \right) \) satisfy (2.3).

Then the mapping \( S_{\beta, \alpha, A} \rightarrow S_{\beta, \alpha, A} \) and \( \psi \rightarrow \mathcal{D}_{\beta} \psi \) is a continuous linear map.

b) Let \( \left( b_q \right) \) satisfy (2.4). Then, the map \( \psi \rightarrow \mathcal{D}_{\beta} \psi \) is a continuous linear map into \( S_{\beta, \alpha, A} \).

c) Let \( \left( a_m \right) \) and \( \left( b_q \right) \) satisfy (2.3) and (2.4), respectively. The operation \( \psi \rightarrow \mathcal{D}_{\beta} \psi \) maps \( S_{\beta, \alpha, A} \) into \( S_{\beta, \alpha, A} \) continuously.

Proof of (a) Let \( \psi \in S_{\beta, \alpha, A} \). With the aid of (3.3),

\[ x^\alpha D^\beta \left( \mathcal{D}_{\beta} \psi \left( x \right) \right) \leq 4 x^\alpha D^{m-3} \left( 2 \beta \right) \left( 2 \beta - 1 \right) x^{2\beta-2} D^3 x^\beta \psi \left( x \right) \]
Theorem 4.1 Let $\alpha, \beta$ be real numbers, $\alpha - \beta \geq -1/2$. Let the sequence $(a_m)$ satisfy (2.3) of [5]. The transform is an automorphism from $pS_{\beta,a_m} \rightarrow pS_{\beta,a_m}.AT$. 

Proof: Using (4.1) leads to

$$ \left| y^mD_x^q \left( h_{\alpha,\beta} \psi \right)(x) \right| $$

$$ = \left| (-1)^q \int_0^\infty x^{2^q-1} \alpha+q, \beta+q (xy) \phi(x) dx \right| $$

$$ \leq C_{m,q,\alpha,\beta} \int_0^\infty x^pD_x^q \phi(x) dx $$

$$ \leq C_{m,q,\alpha,\beta} \sum_{k=0}^{p+2} x^k D_x^q \phi(x) $$

$$ \leq C_{m,q,\alpha,\beta} \sum_{k=0}^{p+2} \int_0^\infty x^{p+3} \alpha+q, \beta+q (xy) \phi(x) dx $$

$$ \leq C_{m,q,\alpha,\beta} \left( p+3 \right) (A+\delta)^2 ST^2 \alpha_2 \left( (A+\delta)T \right)^\alpha a_p $$

for some constant $C'_{m,q,\alpha,\beta}$. Implies $h_{\alpha,\beta} \in pS_{\beta,a_m,AT}$. Thus the proof.

4 GENERALIZED HANKEL-CLIFFORD TRANSFORMATION OF TEMPERED ULTRADISTRIBUTIONS

The generalized Hankel Clifford transformation of the duals of Roumieu type tempered ultradistributions is defined as the adjoint operator.

$$ D_x \left( x^{\beta} \left( a_{\beta} (xy) \right) \right) = x^{\beta} \left( a_{\beta+1} (xy) \right). \quad (4.1) $$
\[ |y^m D^q_y \psi (h_{a,\beta} \psi)(x)| \leq C_{m,q,a,\beta} \sum_{k=0}^{p+2} C_q^\beta (A+\delta)^k (B+\rho)^q b_q a_k |y^m D^q_y \psi (h_{a,\beta} \psi)(x)| \leq C_{m,q,a,\beta} C_q^\beta (p+3)(A+\delta)^2 \times S T^2 a_2 \left( (A+\delta)^r (B+\rho)^q b_q a_p \right) \leq \sigma_{m,q,a,\beta} (A+\delta)^r (B+\rho)^q b_q a_p \]

where
\[ \sigma_{m,q,a,\beta} = C_{m,q,a,\beta} C_q^\beta (p+3)(A+\delta)^2 S T^2 a_2. \]

Thus the proof.

References


